**Q1. [8 marks]**

Let $X_1, X_2, \ldots, X_n$ b a random sample from a uniform distribution over the interval $(\alpha, \beta)$.

i) Find the **method of moments estimators** $\tilde{\alpha}$ and $\tilde{\beta}$ of $\alpha$ and $\beta$. **[4 marks]**

ii) Are $\tilde{\alpha}$ and $\tilde{\beta}$ **consistent** estimators of $\alpha$ and $\beta$, respectively. **[4 marks]**

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**Q2. [8 marks]**

Let $X_1, X_2, \ldots, X_n$ b a random sample from a distribution with p.d.f.

$$f(x) = e^{-(x-\theta)}, \qquad x \geq \theta.$$

Which of the two estimators $\hat{\theta}_1 = \overline{X} - 1$ and $\hat{\theta}_2 = Y_1 - \frac{1}{n}$, where $Y_1$ is the smallest order statistic, is **more efficient**?

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**Q3. [8 marks]**

Let $X_1, X_2, \ldots, X_n$ be a random sample from normal distribution with **known** mean $\mu$ and **unknown** variance $\tau$.

i) Show that the **maximum likelihood estimator** of $\tau$ is $\hat{\tau} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$. **[3 marks]**

ii) Is $\hat{\tau}$ an unbiased estimator of $\tau$ ? **[2 marks]**

iii) Is $\hat{\tau}$ an efficient estimator of $\tau$ ? **[3 marks]**

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**Q4. [8 marks]**

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with p.d.f.

$$f(x) = \frac{1}{\sqrt{\pi \beta}} x^{-\frac{1}{2}} e^{-x/\beta}, \qquad x > 0, \quad \beta > 0.$$

i) Is $V = \frac{2}{\beta}\sum_{i=1}^{n} X_i$ a **pivot** for $\beta$? **[5 marks]**

ii) Given the random sample $x_1 = 2, x_2 = 1, x_3 = 4, x_4 = 3, x_5 = 10$, from the above distribution, find a $90\%$ confidence interval for $\beta$. **[3 marks]**

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**Q5. [8 marks]**

Let $X_1, X_2, \ldots, X_{20}$ be a random sample from the **normal** distribution with mean 0 and variance $\tau$.

i) Find the **best critical region** of size $\alpha = 0.01$ for testing

$H_0 : \tau = 1$ vs. $H_1 : \tau = 0.25$ **[5 marks]**

ii) Show that the **power of the test**, at $\tau = 0.25$, exceeds 0.95. **[3 marks]**

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**Q6. [10 marks]**

Let $X_1, X_2, \ldots, X_{n_1}$ and $Y_1, Y_2, \ldots, y_{n_2}$ be two independent random samples from $N(\mu_1, \tau)$ and $N(\mu_2, \tau)$, respectively, where $\mu_1$, $\mu_2$, $\tau$ are unknown parameters.

Derive the **likelihood ratio test** of size $\alpha$ for testing

$$H_0 : \mu_1 = \mu_2 \qquad \text{vs.} \qquad H_1 : \mu_1 > \mu_2 .$$

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