**Q1. [6 marks]**

Let $X_1, X_2, \ldots, X_n$ be a random sample from a gamma distribution with parameters $\alpha$ and $\beta$.

i) Find the method of moments estimator $(\tilde{\alpha}, \tilde{\beta})$ of $(\alpha, \beta)$. **[3 marks]**

ii) Is $(\tilde{\alpha}, \tilde{\beta})$ consistent estimator for $(\alpha, \beta)$? **[3 marks]**

**Q2. [10 marks]**

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with p.d.f.

$$f(x;\theta) = \theta(1+x)^{-(\theta+1)}, \qquad x > 0, \quad \theta > 0.$$

i) Find the maximum likelihood estimator $\hat{\theta}$ of $\theta$. **[3 marks]**

ii) Is $Y = \frac{n-1}{n}\hat{\theta}$, $n > 1$, unbiased estimator for $\theta$? **[3 marks]**

iii) Using **Rao-Cramer** theorem, is $Y$ efficient estimator for $\theta$? **[4 marks]**

**Q3. [9 marks]**

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with p.d.f.

$$f(x;\beta) = \frac{1}{2\beta^3} x^2 e^{-x/\beta}, \qquad x > 0, \quad \beta > 0.$$

i) Is $V = \frac{2}{\beta}\sum_{i=1}^{n} X_i$ a pivot for $\beta$? **[4 marks]**

ii) If a sample of size $n = 4$ yielded $\sum_{i=1}^{4} x_i = 25$, find a 95% confidence interval for $\beta$.

**[5 marks]**

**Q4. [9 marks]**

Let $X_1, X_2, \ldots, X_n$ be a random sample from a **Poisson** distribution with mean $\lambda$.

i) Find a **complete sufficient statistic** for $\lambda$. **[5 marks]**

ii) Using **Lehmann-Scheffe** theorem, is $T = \left(\frac{n-1}{n}\right)^{\sum_{i=1}^{n} X_i}$, $n > 1$, **minimum variance unbiased estimator** (MVUE) for $g(\lambda) = e^{-\lambda}$. **[4 marks]**

**Q5. [6 marks]**

Let $X_1, X_2, \ldots, X_{100}$ be a random sample from a distribution with mean $\mu$ and variance $4$. To test

$$H_0: \mu = 1 \qquad \text{vs.} \qquad H_1: \mu > 1$$

we consider the **critical region**

$$C = \{ (x_1, \ldots, x_{100}) : \bar{x} \geq k \}.$$

i) Find $k$ such that the **significance level** of the test is approximately $0.011$. **[3 marks]**

ii) Find the **power function** of the test at $\mu = 2$. **[3 marks]**

**Q6. [10 marks]**

Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be two independent random samples from $N(0, \theta_1)$ and $N(0, \theta_2)$ populations, respectively.

Find the **likelihood ratio test** of size $\alpha$ for testing

$$H_0: \ \theta_1 = \theta_2 \qquad \text{vs.} \qquad H_1: \ \theta_1 > \theta_2.$$