1. **(5 pts)** A coin is tossed repeatedly until two successive heads appear. Let X_n be the cumulative number of successive heads.

a) **(2 pts)** Show that $\{X_n,\ n=0,\ 1,\ \ldots\}$ is a Markov chain with state space $S = \{0,1,2\}$ and transition probability matrix:

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{pmatrix}.$$

b) **(1 pt)** Starting with no successive head, what is the probability that we end up with 2 successive heads?

c) **(1 pt)** Starting with no successive head, what is the mean number of tosses required to get 1 successive head prior to ending up with 2 successive heads?

d) **(1 pt)** Starting with no successive head, what is the mean number of tosses required to get 2 successive heads?

2. **(15 pts)** A particle moves among the states 0, 1, 2 according to a Markov chain (MC) $\{X_n,\ n=0,1,\ldots\}$ with the transition probability matrix:

$$\mathbf{P} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0.0 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$

The initial distribution for the particle is $P[X_0 = 0] = 0.4$, $P[X_0 = 1] = 0.5$, and $P[X_0 = 2] = 0.1$.

a) **(1 pt)** Specify the communicating classes.

b) **(1 pt)** Classify each state as either transient or recurrent.

c) **(1 pts)** Find the period of each state.

d) **(2 pts)** Specify whether the MC is (i) irreducible, (ii) aperiodic, (iii) positive recurrent, and (iv) regular.

e) **(4 pts)** Compute:

(i) $Pr[X_0 = 1, X_1 = 2, X_2 = 1]$

(ii) $Pr[X_2 = 2, X_3 = 2, X_4 = 1 | X_1 = 0]$

(iii) $Pr[X_3 = 2 | X_1 = 1]$

(iv) $Pr[X_2 = 2]$

f) **(2 pts)** What fraction of time, in the long run, is the particle in state 0? How about states 1 and 2?

g) **(1 pt)** Compute $f_{11}^{(3)}$, the probability that starting from state 1, the first return of the particle to state 1 will be at the 3rd move.

h) **(1 pt)** Compute $f_{02}^{(3)}$, the probability that starting from state 0, the first passage of the particle to state 2 will be at the 3rd move.

i) **(1 pt)** Compute m_1, the expected number of moves required that the particle leaves state 1 and returns to state 1 for the first time.

j) **(1 pt)** Compute m_{02}, the expected number of moves that the particle goes through to get from state 0 to state 2 for the first time.

3. **(6 pts)** Suppose that X_1, X_2, and X_3 are independent random variables with $f_{X_i}(x) = \lambda_i e^{-\lambda_i x}$, $x > 0$, i=1,2,3, where $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$, respectively.

a) **(2 pts)** If $Y = min\{X_1, X_2, X_3\}$, show that $P(Y < 1/6) = 1\text{-}e^{-1}$.

b) **(2 pts)** Compute $P(X_2 > 3 | X_2 > 1)$.

c) **(2 pts)** Show that $P(X_1 < X_3) = 1/4$.

4. **(3 pts)** Suppose $P(Y = k | X = x) = \dfrac{e^{-x} x^k}{k!}$, $k = 0, 1, \ldots$, where $f_X(x) = \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\beta x}$, $x > 0$.

a) **(1 pts)** Compute $E(Y)$.

b) **(2 pts)** Compute $E(Y^2)$.

5. **(4.5 pts)** Suppose that customers arrive at a station according a non-homogenous Poisson process with the mean rate function:

$$\lambda(t)=\begin{cases} 4t, & if\ 0\le t<1 \\ 4, & if\ 1\le t<3 \\ -4t+16, & if\ 3\le t\le 4, \end{cases}$$

where t is in hours.

a) **(1.5 pts)** What is the probability that at least 1 customer arrive to the station during the first hour?

b) **(1.5 pts)** Given 2 customers arrive in the first hour, what is the probability that there will be 4 customers in the first 2 hours?

c) **(1.5 pts)** What is the probability that 1 customer arrives in the first hour, 2 customers in hours 1 to 3, and 3 members in hours 3 to 4?

6. **(6.5 pts)** Patients arrive to a medical clinic according to a Poisson process $\{X(t), t \ge 0\}$ with rate 4 per hour. The clinic has two doctors. Suppose that 25% of arriving patients are randomly sent to doctor 1 according to process $\{X_1(t), t \ge 0\}$, and the remaining 75% are sent to doctor 2 according to process $\{X_2(t), t \ge 0\}$.

a) **(1.5 pts)** Given 4 patients arrive to the clinic during the first two hours, what is the probability that one patient arrives in the first 30 minutes, one in the next hour, and the remaining two arrive in the last 30 minutes?

b) **(5 pts)** Compute:

(i) $E[\{X(1)\}^2]$

(ii) $E[X(1)X(3)]$

(iii) $E[X(3)|X_2(3)=2]$

(iv) $Pr[X(1)\geq 2|X(3)=4]$

(v) $Pr[X_1(1)=1|X(2)=3]$